Polytiles: Equilateral and Equiangular Polygons (part 1a)

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Abstract

This introductory paper defines a polytile or p-tile as an equilateral polygon with turn angles as multiples of 360°/p, for even p=4, 6, 8…. Polytile notation is offered as a simple way to describe these polygons, with a geometric interpretation as a cycle of edge-to-edge regular p-gons, with vertices centered on each p-gon. A wide range of examples polygons and tilings are shown in this paper to demonstrate a breadth of value. A next paper, part 1b, will formalize the definition and report an enumeration of convex polytiles. Further papers will elaborate topics introduced in this paper including stars, compounds, extending notations, recursively extended polygons, symmetry, naming of polygons, and partial and fractional polytiles which allow one or two unequal edge lengths.

1. Introduction

Equilateral polygons are polygons with equal edge-lengths. **Equiangular polygons** [2] are polygons with equal vertex angles. Polygons which are both equilateral and equiangular are called **regular**. This paper primarily looks at equilateral polygons, with section 5 exploring equiangular ones. When we are looking at *equilateral polygons*, we will count colinear vertices as vertices, while for *equiangular polygons*, we will ignore these and count as longer edges with integer lengths.

The simplest irregular *equilateral polygons* are **isotoxal** [12] or *edge-transitive*, with identical edges within its symmetry, alternating two vertex types. The simplest *equiangular polygons* are **isogonal** or *vertex-transitive*, with identical vertices within its symmetry, alternating two edge types. Figure 1 shows hexagons: regular, isotoxal, and isogonal polygons. Beyond isotoxal and isogonal polygons, the degrees of freedom increase. How can we explore these?

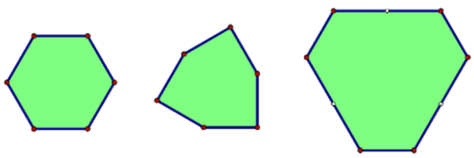

Figure 1: Regular hexagon, isotoxal (equilateral), and isogonal (equiangular)

The simplest general way to describe a polygon is a directional path of straight movements and turns that returns to the starting point. This is how **Turtle Graphics** [1] works, expressed in the programming language Logo that move a virtual turtle across a screen having a position and facing, using commands to turn angles and move straight distances.

Polytiles work in a similar way, with intentional limitations. **Equilateral polygons** start with identical length edges, while polytiles are described as a sequence of turns as **rational divisors** of 360°. A *p*-tile has angles as **integer multiples** of 360°/*p*. Similarly **equiangular polygons** start with equal angles, described as a sequence of **integer edge lengths**. These constraints allow a wide breadth of interesting polygons to be explored.

1.1 Background

I began this exploration in August 2020 using the Java App **Tyler** [11], which allows regular polygons to be connected edge-to-edge. I found I could create cycles of dodecagons that made different shaped gaps. The gaps can be seen as equilateral polygons by drawing vertices centered on each dodecagon and edges between adjacent dodecagons. The dodecagons limit the angles to multiples of 360°/12 or 30°. Figure 2 shows an irregular tiling of green dodecagons which define various convex equilateral polygons. I called these **dodecatiles** and I wondered how many convex ones there were. Looking ahead, figure 6 shows the answer to be 16 unique convex solutions!

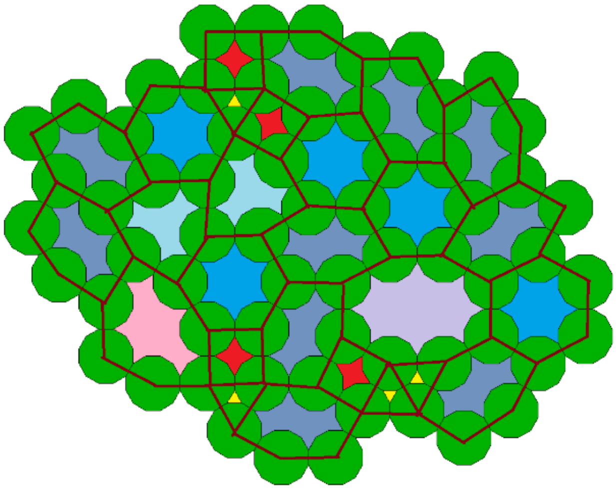

Figure 2: Convex equilateral polygons (outlined) from edge-to-edge dodecagons

1.2 Polytiles

A **polytile or p-tile** is defined with this paper as an equilateral polygon with angles as integer multiples of 360/*p*°. We limit *p* to evens, 4,6,8.... Angles are measured as turn angles, or external angles, supplement to internal angles. The advantage of turn angles is straight (colinear) edges have 0° turns, and concave angles are negative.

All polytiles can be described in **polytile notation p:$a_1.a_2\ldots a_m$^n**, with **p:** defining the **p**-tile, and a_i giving turn angle indices at sequential vertices, with each as a_i/p fractional turns. The ***n*** represents cyclic symmetry of the polygon, repeating all angles of the expression.

If **p:** is not given, the polytile can be assumed to be simple, *computed as* $p=n(a_1+a_2+\ldots+a_m)$.

Turn angles are constrained to the range of ±180°, with the limits at 180° degenerate, best avoided. For a **simple polygon**, the sum of all the vertex turn angles will be 360°, or 1 full turn. In section 4 **star polygons** [1] are explored, which have an integer number of full turns, besides 1. The **turning number** of a polygon is also called its **density**. [5]

In polytile notation, the tetratile **4:1.1.1.1** represents a square, with 4 ¼ turn. It can also be written as **4:1^4**.

Figure 3 shows a concave example with octagons (an octatile) which are labeled and colored with the vertex angle. Its polytile notation is **8:-2.3.0.1.2.1.0.3**, sequencing the 8 vertices going counter clockwise around the polygon, although in this case, with reflection symmetry both directions give the same result.

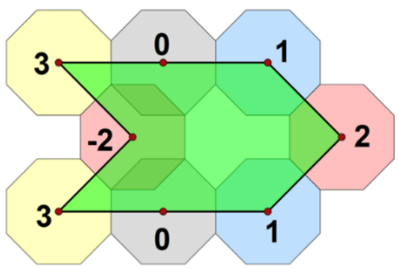

Figure 3: Example octatile

2. Complete sets of convex polytiles

A first question of interest is how many (strictly) convex equilateral polytiles there are, and this can be enumerated for each p. We know there will be infinitely many solutions as soon as we allow colinear edges or concave forms.

A simple exhaustive search can count the number of solutions for small p since the highest convex p-tile has *p* sides, and turns are limited to p/2-1. Rotational duplicates are removed by shifting the starting index, and chiral duplicates are removed, identified by reversing the order.

Part 1b will give a more detailed survey of strictly convex polytiles up to 30-tiles. For a simple enumeration, the number of solutions, (counting chiral pairs as the same), are given in table 1.

Table 1: Counts for strictly convex p-tiles

p-tile	4	6	8	10	12	14	16	18	20	22	24	26	28	30
Count	1	3	4	7	16	17	28	70	85	125	392	379	704	3359

Figure 4 shows the strictly convex solutions for tetratiles (4-tiles), hexatiles, (6-tiles), octatiles (8-tiles). Figure 5 shows decatiles (10-tiles), and Figure 6 shows dodecatiles (12-tiles).

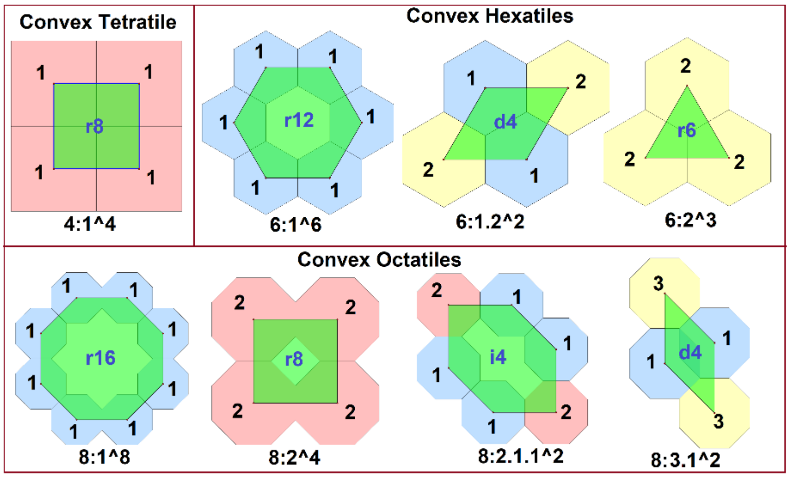

Figure 4: One tetratile, three hexatiles and four octatiles

For **p=4 (tetratiles)**, the only solution is the square, expressed as **4:1.1.1.1** or **4:1^4**. See examples in section 2.2 of concave tetratiles.

For **p=6 (hexatiles)**, we find 3 solutions: a regular hexagon **6:1^6**, rhombus **6:1.2^2**, and equilateral triangle, **6:2^3**. All three are monohedral tilings, able to self-tile the plane. See examples in section 2.3 of concave hexatiles.

For **p=8 (octatiles)**, there are 4 solutions: regular octagon **8:1^8**, square **8:2^4**, a hexagon **8:2.1.1^2** and rhombus **8:3.1^2**. You can see the tetratile square, **4:1^4**, is repeated as octatile **8:2^4**. The square and rhombus of the octatiles are used in **Ammann aperiodic tilings**, set A5. [15] See examples in section 2.4 of concave octatiles.

For p=10 (**decatiles**), there are 7 solutions, shown in figure 5: regular decagon, an octagon, 2 hexagons, a regular pentagon, and 2 rhombi.

The two decatile rhombi are used in the Penrose aperiodic tile set of rhombi P3. [14] See examples in section 2.5 of concave decatiles.

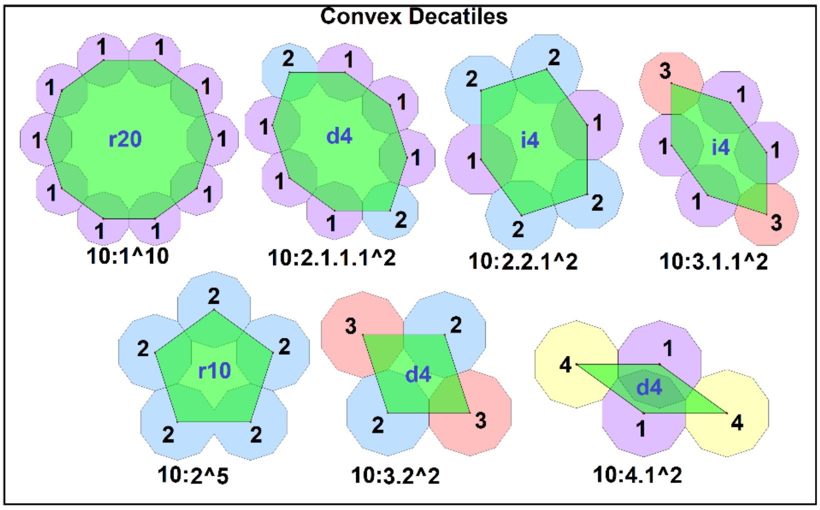

Figure 5: Set of convex decatiles

Note: To help differentiate the equilateral solutions, the symmetry of each convex polytile is given in its center, using **Conway polygon symmetry** [4], with a letter followed by group order. (r=regular, g=gyro, d=diagonal mirrors crossing vertices, p=perpendicular mirrors crossing edges, i=iso-symmetry with mirrors in both vertices and edges.)

For p=12 (**dodecatiles**), there are 16 solutions, shown in figure 6.

These include a first chiral pair (boxed together). Vertices given counterclockwise. Five of the convex dodecatiles are used in Pattern Blocks. [18] See example concave dodecatiles in section 2.6.

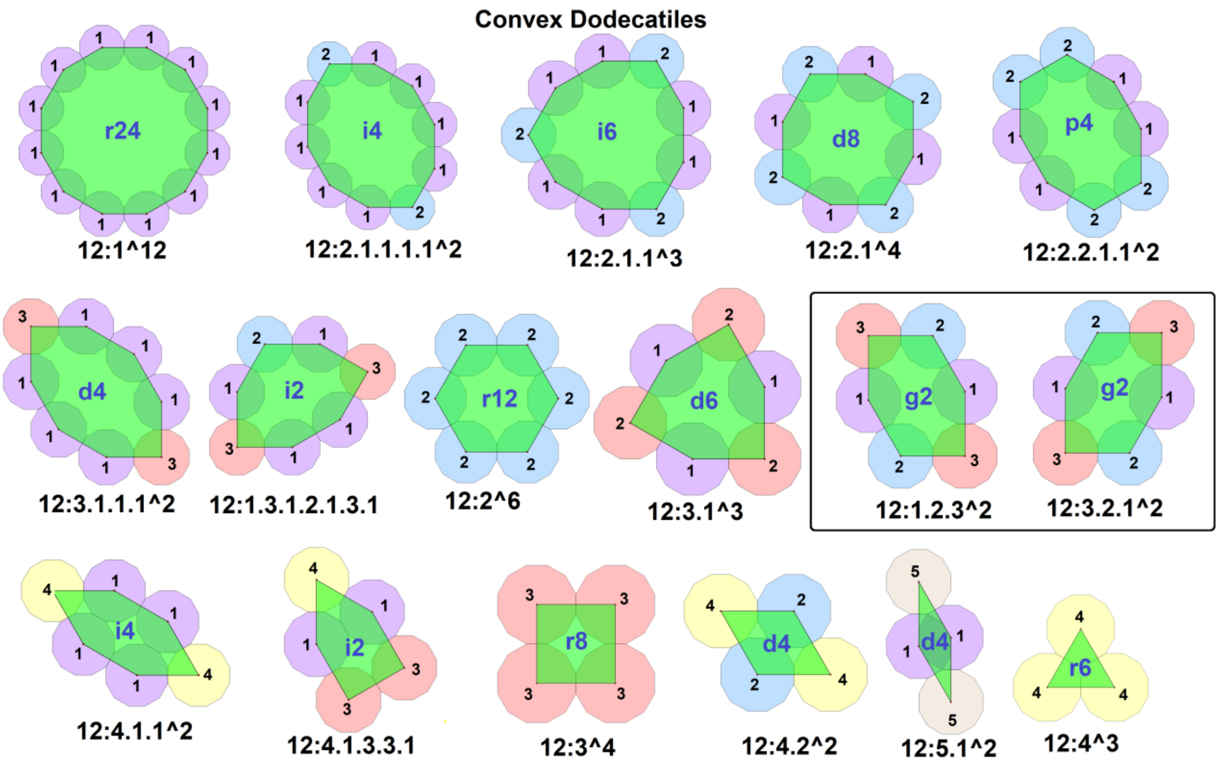

Figure 6: Set of convex dodecatiles

2.1 Special properties of equilateral polygons

Equilateral polygons have several useful properties:

A first property is **dissection** [6]. An equilateral may be dissected into smaller equilaterals. Figure 7 shows the 16 dodecatiles dissected into triangles, squares, and narrow rhombi.

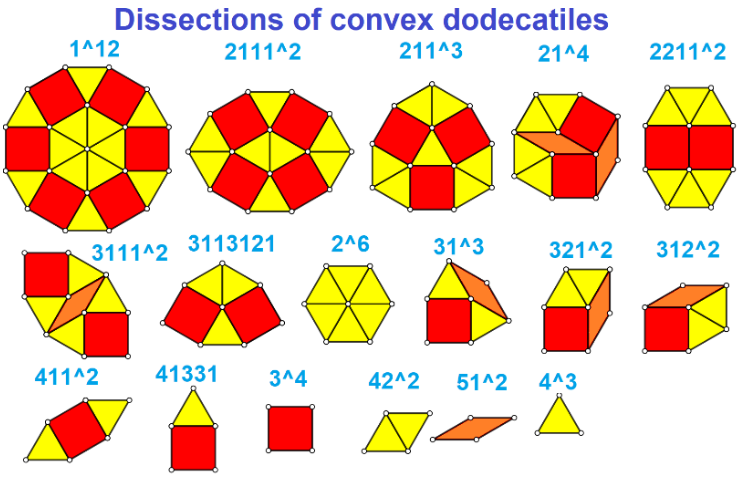

Figure 7: Convex dodecatiles dissections

A similar property in reverse, **attachment**. a new polytile can be defined by attaching two or more polytiles edge-to-edge with the external perimeter as a new boundary.

Thirdly any simple domain bounded with a cycle of edge-to-edge cycle of equilateral tiles is a new equilateral tile, a **gap** or **hole** tile.

These properties allow new equilateral tiles to be more quickly defines from existing ones. Dissection can be used as a constraint to enumerate certain sets of concave tiles, by starting with a convex equilateral tile and removing smaller convex equilateral tiles.

For example, figure 8 shows an example hand-searched set of partial dissections of an octagon. Flat hexagon partial dissections are listed first and the cases are ignored on the full octagon enumeration. The red domains allow complete convex dissections, while the lower set of blue cases are "wild", cannot be complete dissected by the smaller convex octatiles.

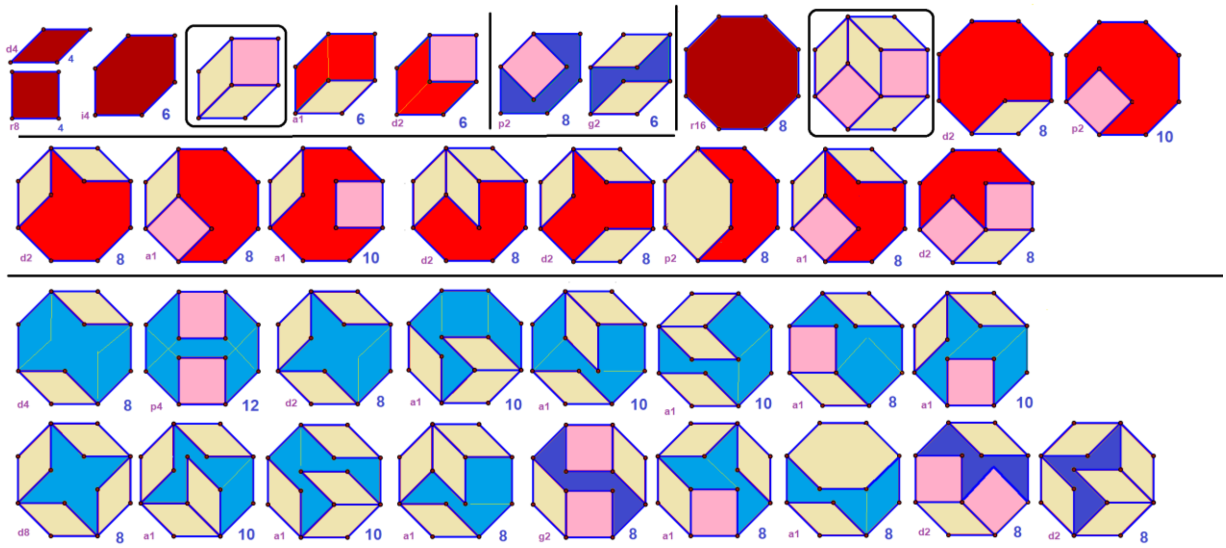

Figure 8: Partial dissections of a regular octagon, using squares and rhombi

2.2 Tetratiles (90°)

Tetratile polygons can fill the plane along the boundaries of a square grid, with angles 90°. All tetratiles have their vertices and edges along the vertices and edges of square graph paper.

Polyominos are edge-to-edge connections of squares, used in puzzles to fill open or closed domains in the plane. [10]

Figure 9 shows seven **Tetrominos** made from 4 squares.

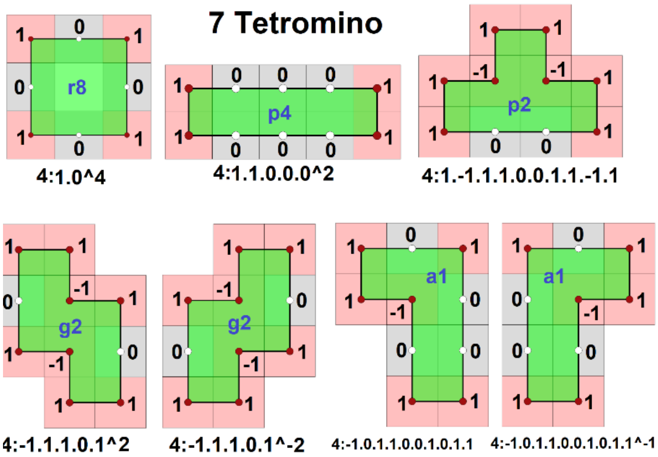

Figure 9: Seven tetrominos with polytile notation

Note: Two tetrominos exist in chiral pairs, which can be differentiated in polytile notation by opposite sign exponents, allowing ^-1 for the chiral L tile. The negative exponent simply reverses the order of the angles listed.

Figure 10 shows an example tiling with 2 concave tetratiles. The 1 (convex) turning vertices fit into the complementary -1 (concave) vertices.

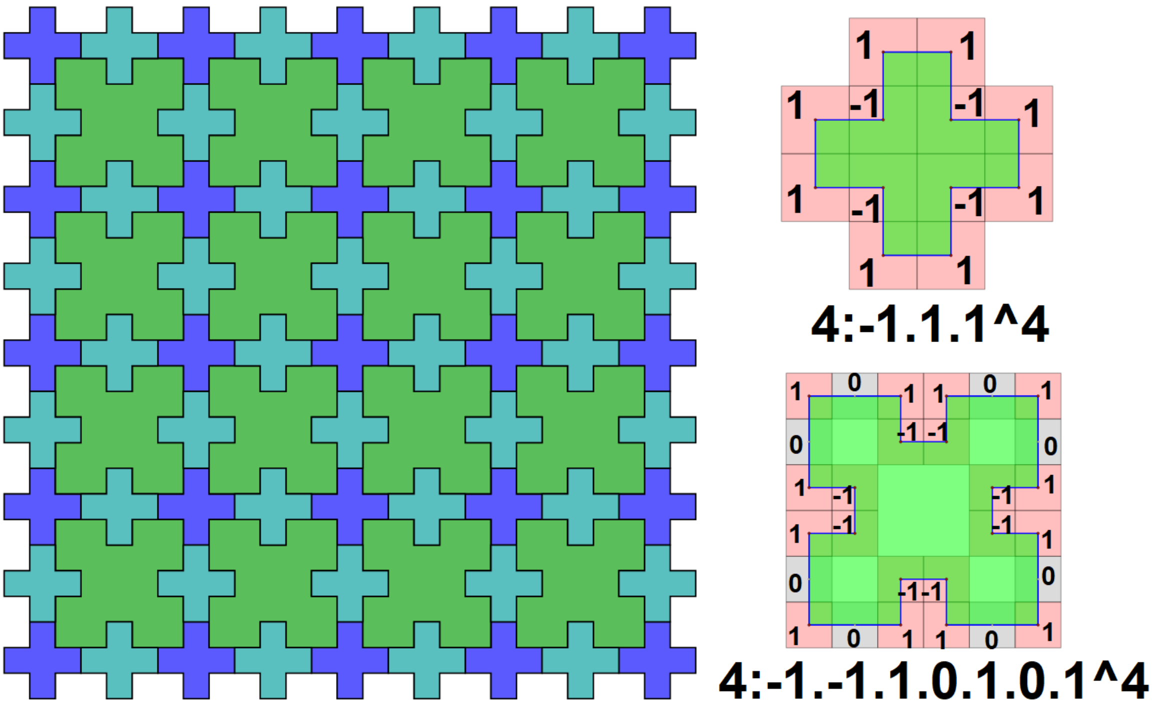

Figure 10: A tetratiling with 2 tiles

2.3 Hexatiles (60°)

Hexatiles are based on 60° angles of a regular hexagon. They can be drawn on triangular graph paper and be any size.

A **polyiamond** is a hexatile constructed by the union of edge-to-edge equilateral triangles. [9]

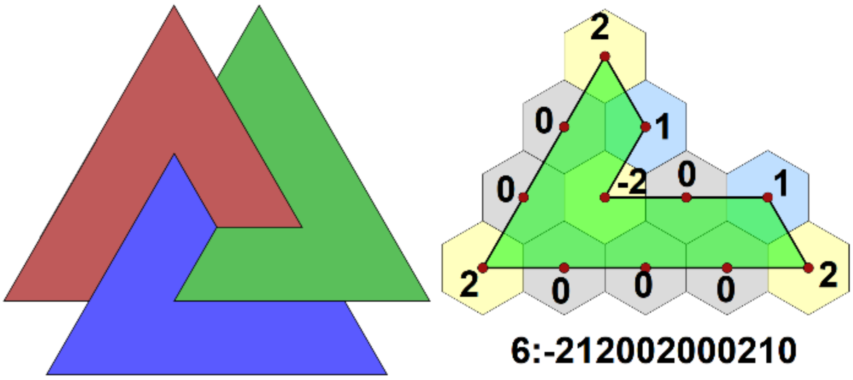

Figure 11: A trefoil pattern with 3 tiles

Figure 11 shows an example partial tiling made with 3 copies of a single polyiamond.

Note: You can see for more brevity the dot delimiter may be suppressed for single digit turns. If there are no delimiters given, it can be assumed to be composed of single digits.

Figure 12 shows a single tri-spiral tile that can tile the plane. [7] It uses long sequences of zeros to express longer straight edges.

Polytile notation is 6:110100100010000100000-10000-1000-100-10-1-12^3, with no delimiter characters.

Note: Repeated angles, here 0s, can be optionally collected into sub-expressions, 6:11010010001(0^4)1(0^5)-1(0^4)-1000-100-10-1-12^3, which is a little shorter.

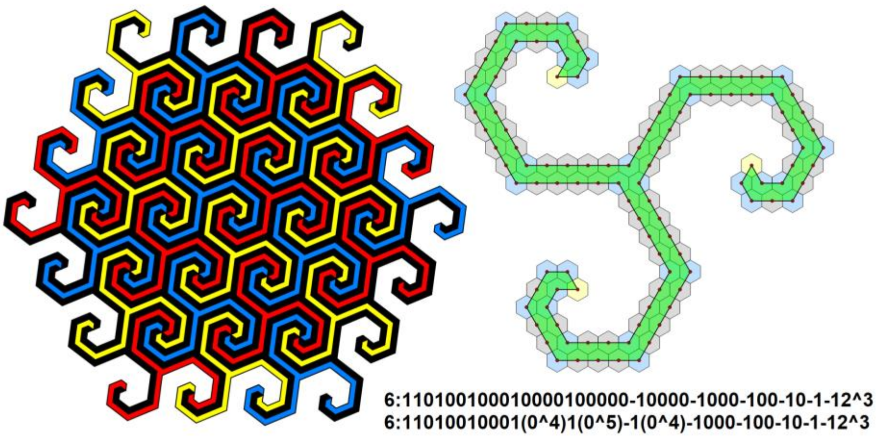

Figure 12: A monohedral hexatiles tri-spiral

2.4 Octatiles (45°)

Octatiles are based on 45° angles of a regular octagon.

Figure 13 shows two tilings by the 4 octatiles. The octatiles are also demonstrated right in plastic disks toys, called Brain Flakes [17], having slots in 8 directions. They can reproduce any octatile, with half of the disks in the plane as "vertices", and half perpendicular as "edges".

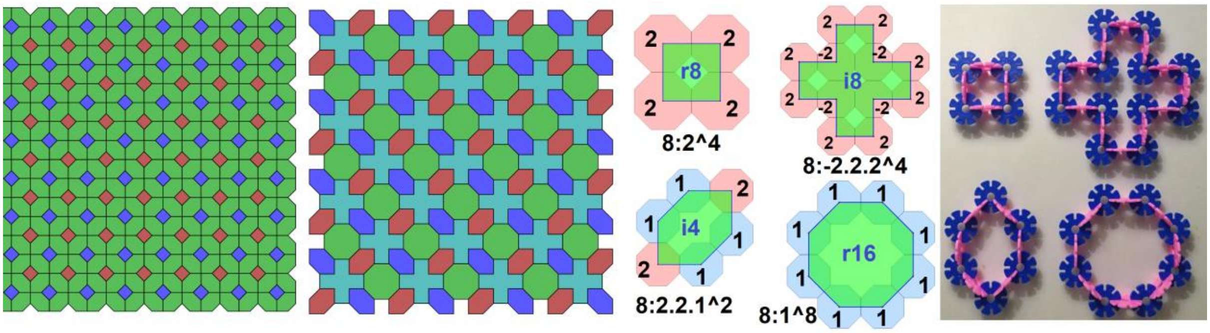

Figure 13: Two tilings made with four octatiles, as "Brain Flakes" (right)

Figure 14 shows a monohedral octatile, a symbolic Starship Enterprise.

Note that some of the edge-to-edge regular octagons overlap. This is fine. It just makes the edge-to-edge octagon visualization a little less helpful.

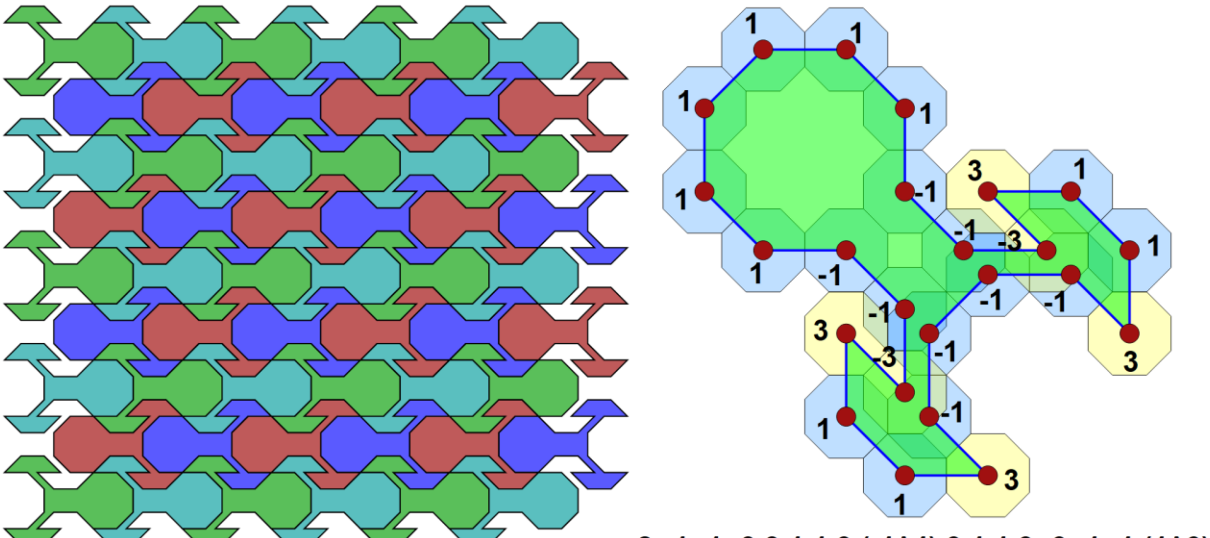

Figure 14: A monohedral octatile

The tile notation for the tile is 8:-1-1-33113(-1^4)3113-3-1-1(1^6) is long while it has bilateral symmetry. A briefer **partition notation** 8:||111-1-1-33113-1-1||, allowing half of the vertices to be specified, starting and ending at lines of reflection.

2.5 Decatiles (36°)

Decatiles are based on 36° angles of a regular decagon.

Figure 15 shows two rosette tilings using the **Penrose aperiodic tile sets P1 and P3**. [14] Set P1 has a pentagon **10:2^5**, star **10:-2.4^5**, a half-star **10:1.1.4.-2.4.-2.4**, and narrow rhombus **10:1.4^2**. Set P3 includes the narrow rhombus and a wide one 10:2.3^2.

Note: **Penrose set P2** includes kites and darts which are not equilateral, but section 6.2 shows how these can be defined as center-fractional polytiles.

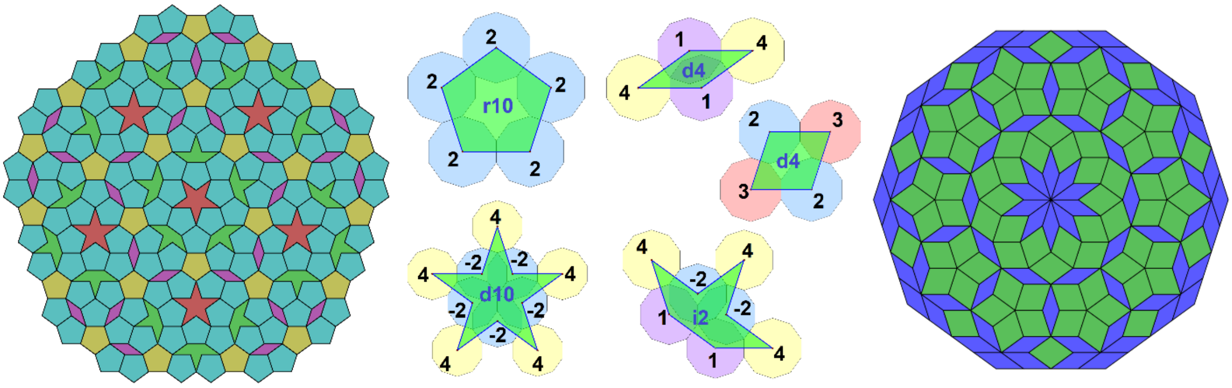

Figure 15: Two Penrose tilings, from sets P2 and P3

In figure 16, we have a flattened octagon and a rhombus tiling. Away from the central pentagon, all the vertices contain two octagons and one rhombus, similar the truncated square tiling with 2 octagons and one square at each vertex, but this variation has 5 orientations of the decatiles.

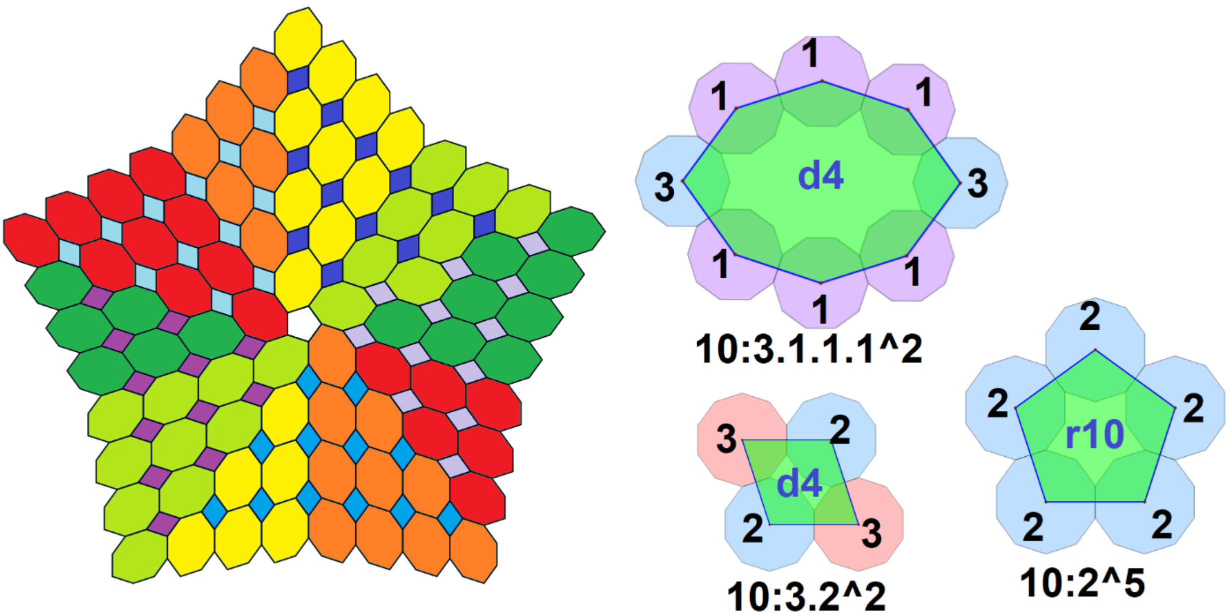

Figure 16: A decatile with an octagon and rhombus in 5 orientations

2.6 Dodecatiles (30°)

Dodecatiles are based on 30° angles of a regular dodecagon. Some of the convex dodecatiles exist as **Pattern Blocks**. [18]

Figure 17 shows a Pattern Block tiling that uses 5 tile types, including a trapezoid which is convex, but not strictly convex, having 2 colinear edges.

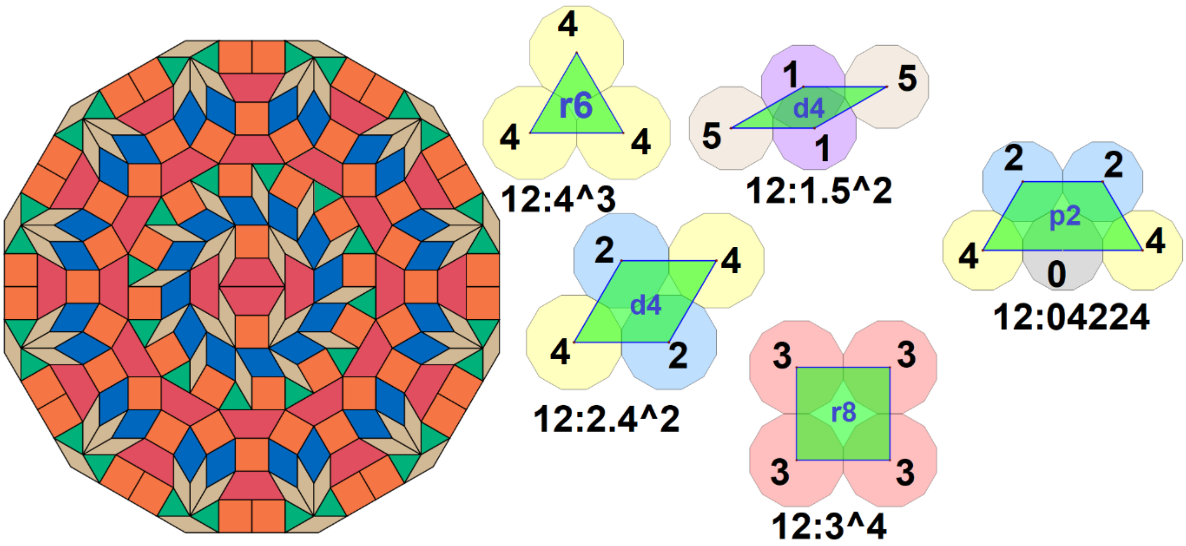

Figure 17: A Pattern Block creation using 5 dodecatile

Figure 18 shows crescent dodecatiles are positioned around the perimeter of the 16 convex dodecatiles. The blue crescents contact the central polygon edges. Each crescent can be dissected into 2 narrow rhombi, 2 wide rhombi, and a square.

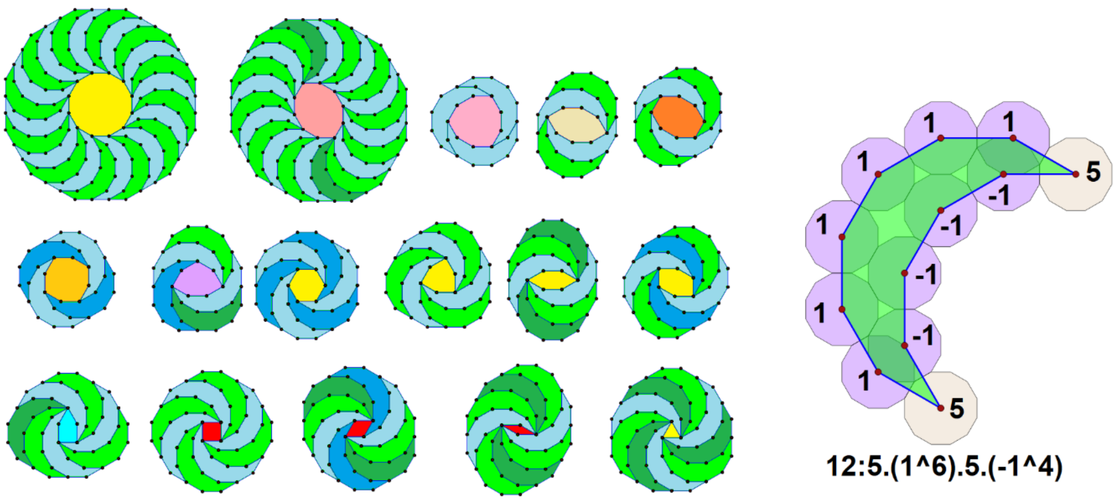

Figure 18: Crescent dodecatiles surrounding 16 convex dodecatile

2.7 Tetradecatile (25 5/7°)

A tetradecatiles is based on the 24 5/7° angles of a regular 14-gon.

Self-tilings can also exist from a single point, like this reflexed heptagon in Figure 19. [13]

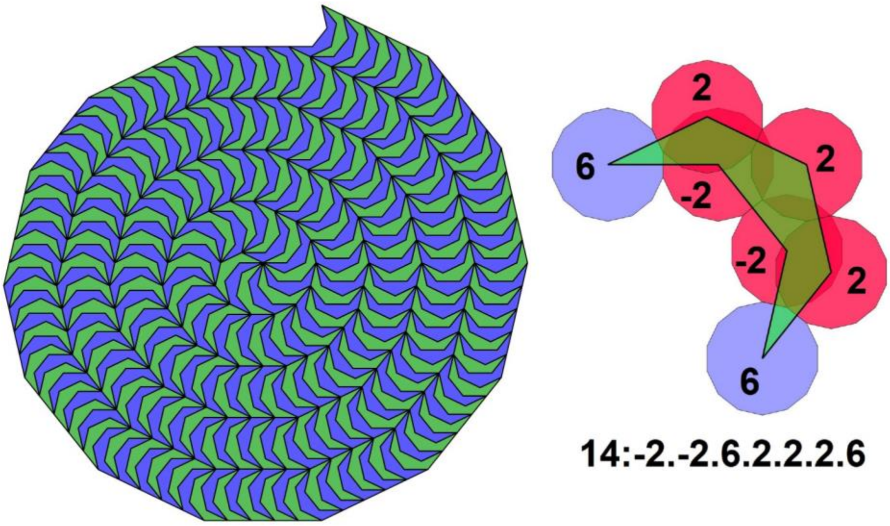

Figure 19: A reflexed heptagon as a monohedral 14-tile

2.8 Octadecatiles (20°)

An octadecatile is based on the 20° angles of a regular 18-gon.

Figure 20 shows Michael Hirschhorn's nonperiodic tiling with an asymmetric pentagon, angles 140°, 60°, 160°, 80°, and 100°. [16] As turn-angles (40°, 120°, 20°, 100°, 80°), this can be reduced by a factor of 20, into an octadecatile: 18:2.6.1.5.4. Figure 20 shows a partial tiling with colors for chiral copies.

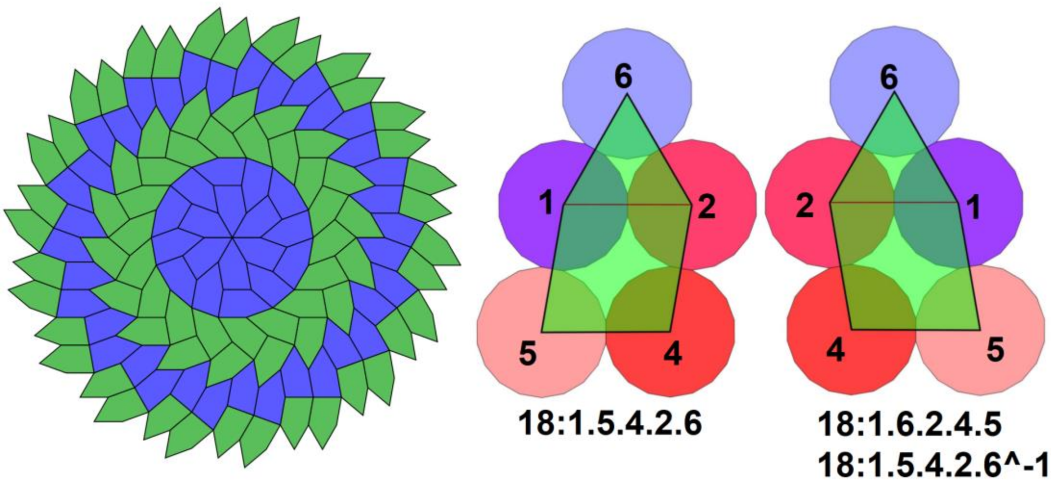

Figure 20: Hirschhorn's monohedral pentagonal tiling as an 18-tile

3. Parametric polytiles

One trick of polytile notation is that using indices allows similar shaped tiles with different angles and symmetry. Figure 21 shows polytile notation expressing **parametric polytiles** of increasing symmetry by having a parameter in the exponent. The top "cog" set change the p-tiles. The notation 4p:(2-p).(2-p).p.p^p has vertices that sum to 4p. The outer angles are p/4p, so quarter turns.

The lower circular "fan" tiles preserve 48-tiles, adding longer (1^a) "arcs" as n-slots increases. The 12s are quarter turns for the squared inner slots. The n=8 "fan" exists as toy pieces that connect perpendicularly slot-to-slot in 3D, called *Octons* and *Brain flakes*. [17]

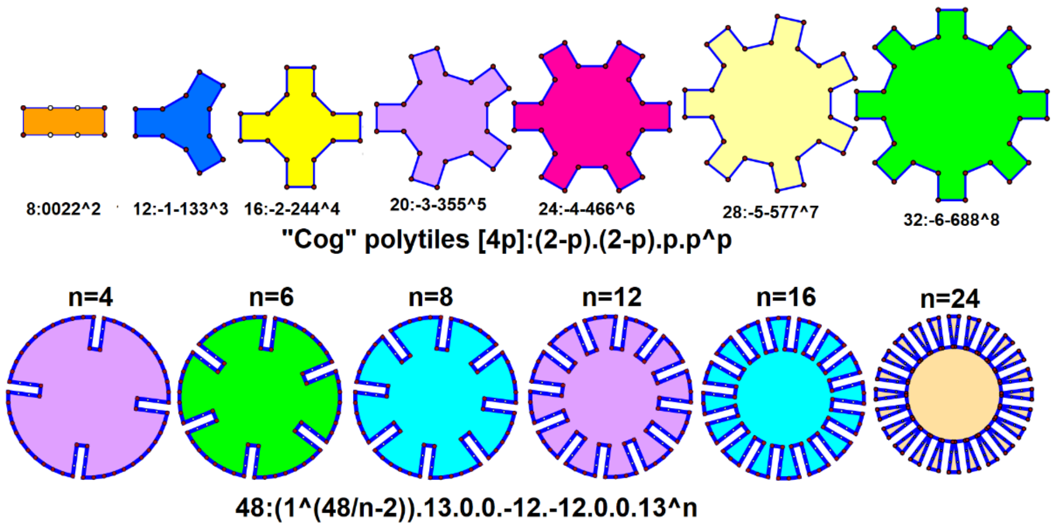

Figure 21: Parametric sets of "cogs" and "fans"

Figure 22 shows the parametric notation -1.2.1.*a*^2, which makes 2(2+*a*)-tiles for different *a*, with 3 cases shown. Despite having similar forms, the self-tiling structures can be quite different.

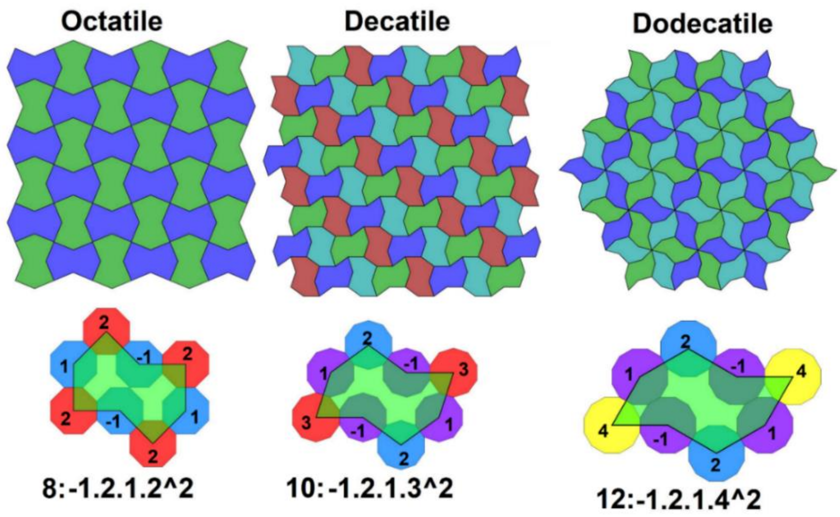

Figure 22: Three similar tiles with different p-tile angle systems

3.1 Polytiles extended by arcs

Polytiles can approximate circular arc-edged tiles. Polytiles can approximate polygons with arc boundaries with sub-expressions of the form *(1^a)* for convex arcs, and *(-1^a)* for concave arcs.

Figure 23 shows 3 24-tiles filling the plane, a wavy triangle, and concave hexagon, and a lens shape. Waving arcs can sequence positive and negative arcs as *(1^a).(-1^a)*.

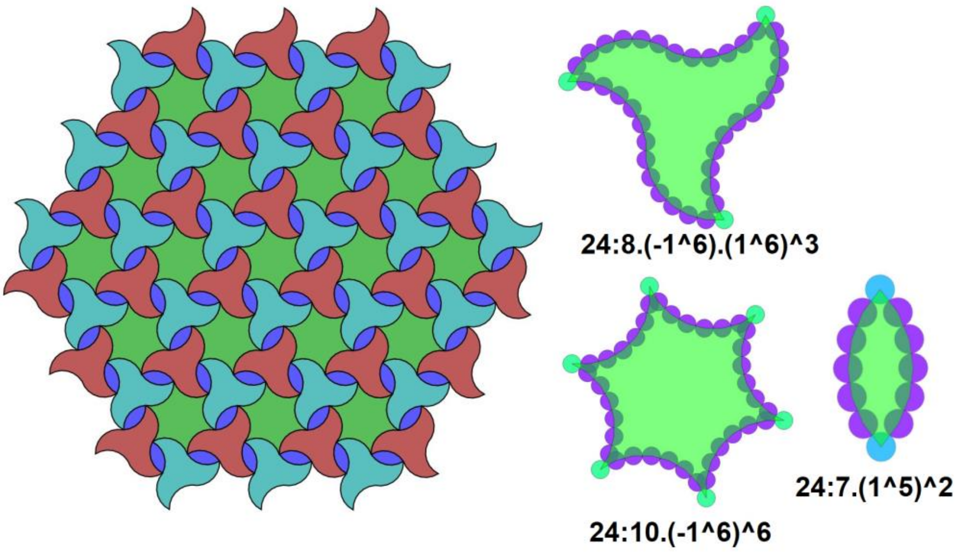

Figure 23: A tiling with arcing polygons

Figure 24 shows a 36-tile approximation monohedral **tricurve** [20] tile, a triangular shape with 1 long convex-arc edge and two concave-arc edges. The example tile has 30°-60°-90° vertex angles and 60°-30°-90° on opposite arcing edge. An a-turn arcing edge is represented by (±1^(*a*-1)). The right angle between the 2 concave arcs has an 8 rather than 9 turn to preserve the turn angle sum of 36.

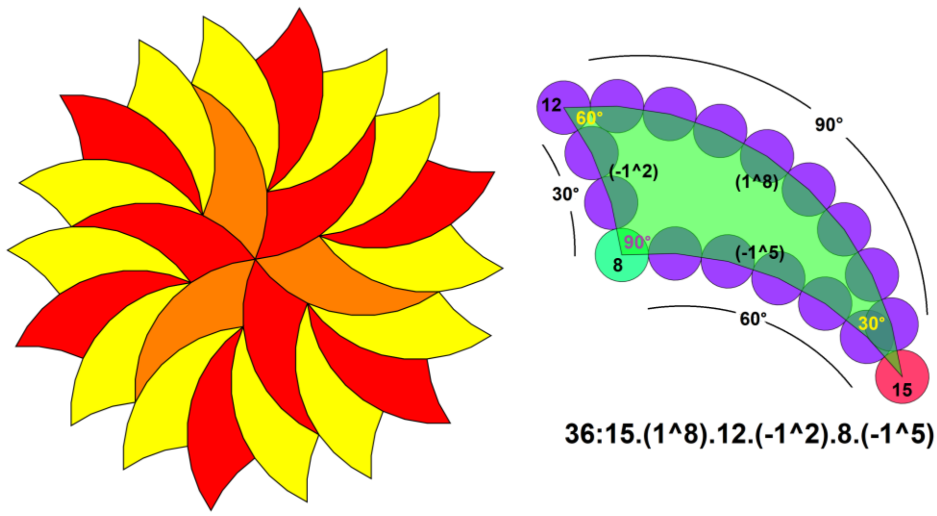

Figure 24: A 36-tile approximating a tricurve monohedral tile

3.2 Polytiles extended by polychains

Polytile notation allows parenthetical subexpressions for patterns, like a zigzag (-1.1^m), and square/trapezoidal patterns (1.1.-1.-1^m), with m-repetitions. We can call these open expressions **polychains**: or specifically **tetrachains** and **dodecachain** below. Linear polychains have their angle sum zero, so they can be added or removed transparently.

Figure 25 shows two examples with tetratiles and dodecatiles. In order to be square the pattern must be ¼ angles, so 1's work for tetratiles, and 3's are needed for ¼ angle dodecatiles.

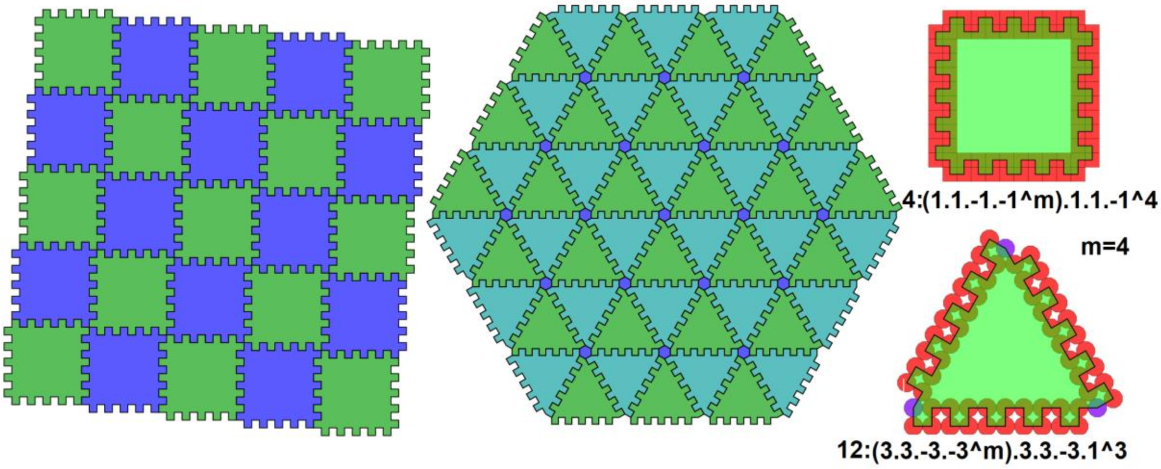

Figure 25: Tilings made with polytiles extended by square waves

3.3 Recursively extended polytiles

Extending polychains (with zero sum) can be applied to any polytile, even recursively.

Figure 26 shows the open polytile pattern 12:0.-3.-3.3.3.0 extending a hexagon (2^6). A second level can be applied by inserting the same pattern alternately into the level 1 extended tile, although this gets unwieldly expressed explicitly. To do recursion well, a powerful extending notation will be introduced in a future paper.

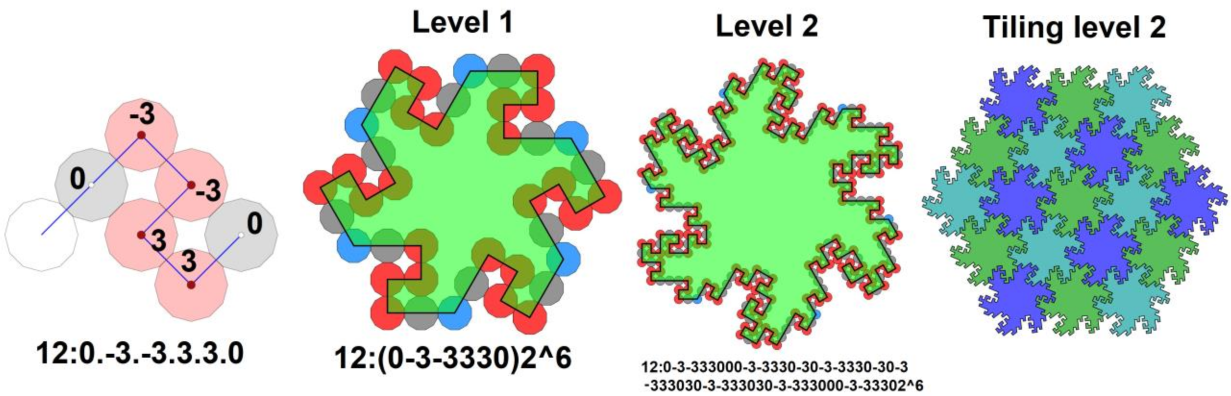

Figure 26: Recursively extended polytiles

4. Star polygons and compounds

Regular star polygons, {p/q} can be made as polytiles "*p*:*q*^*p*", *gcd*(*p*,*q*)=1. The number of turns of a polytile $p{:}a_1\ldots a_m$^n can be computed as $(a_1+a_2+\ldots+a_m)n/p$. *q*=*qp*/*p* for regular star polygons.

Multiturn polytiles cannot be used for tiling, but they are an important class of polygons for exploration. When c=gcd(p,q)>1, the polygon will be degenerate with coinciding copies of vertices and edges *c* times. Those forms can be re-interpreted as compounds, with disjointed cycles, and repeat copies of regular polygon rotated equally spaced, expressed as *c{p/q}*.

Figure 27 demonstrates some examples stars and compounds using decatiles. The top row shows a regular decagon, pentagon, and pentagram along with compound pentagon and pentagrams, are shown, expressed with a *c** prefix.

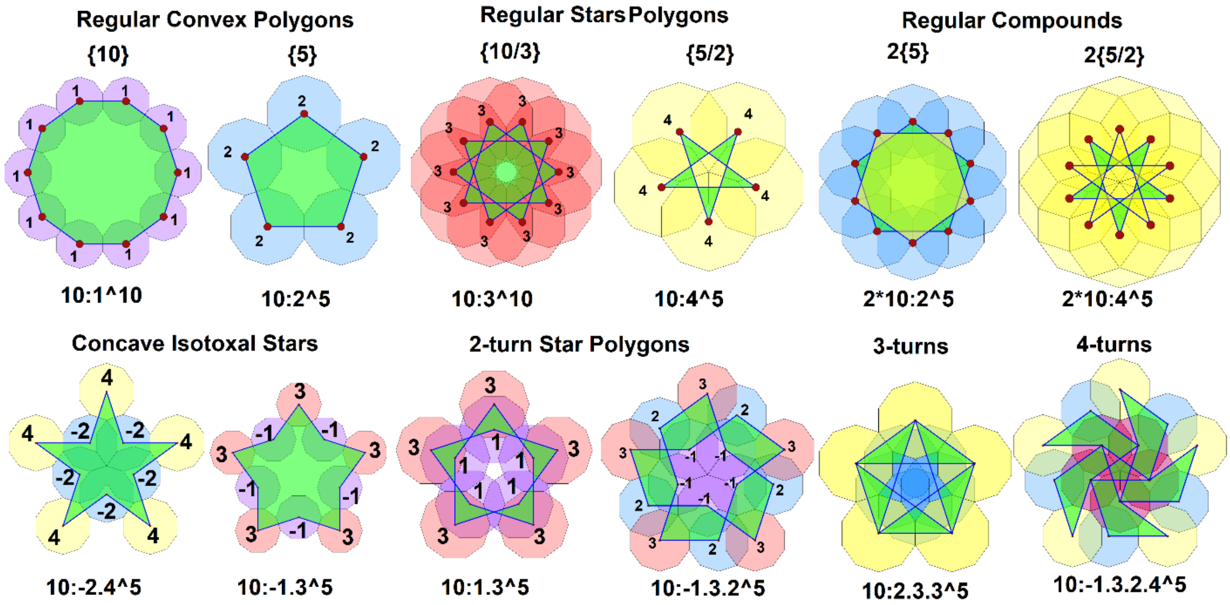

Figure 27: Regular convex, concave and multiturn stars, and compounds

Figure 28 shows four star 24-tiles with arcing boundaries. They are the same form, except adjusting their relative arc lengths and sharp turns.

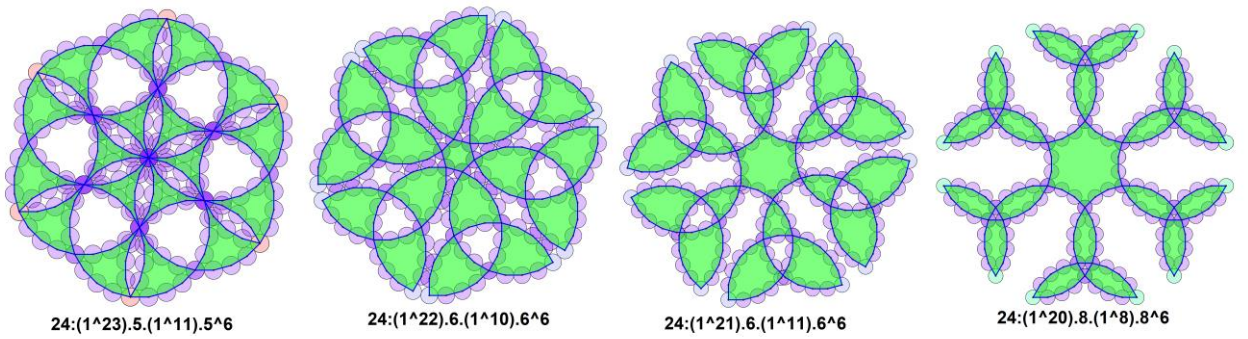

Figure 28: Polytiles with long arcs and sharp turns

5. Equiangular polytiles

Equiangular polygons, those turning the same angle at each vertex, are closely related to regular polygons. A regular polygon has a Schläfli symbol *{p/q}* for p vertices, and q turns around the center. For whole divisor angles, *q*=1, and is suppressed. Each vertex has a turn angle of 360*q/p* degrees, and the sum is 360*q*, so *q* is the number of turns.

All of this is true for equiangular polygons, so they can be given a similar symbol <*p/q*>, choosing angle-brackets as a reminder of equal vertices.

Figure 29 shows an example polytile, an equiangular dodecagon, <12/4>, suppressing colinear vertices and counting them as integer length edges.

Equilateral polytile notation is 6:2.2.0.0.2.2.0.0.0.0^3 or 6:2.2.(0^2).2.2.(0^4)^3.

Equiangular polytile notation can be expressed it more compactly: ***<p/q>:*** $\boldsymbol{e_1.e_2\ldots e_m}$***^n***, where $\mathbf{e_1,e_2\ldots e_m}$ is a sequence of integer edge lengths, with turn angles 360q/p at all vertices. This notation can be translated back to the original p-tile notation by sequential zeros for longer edges.

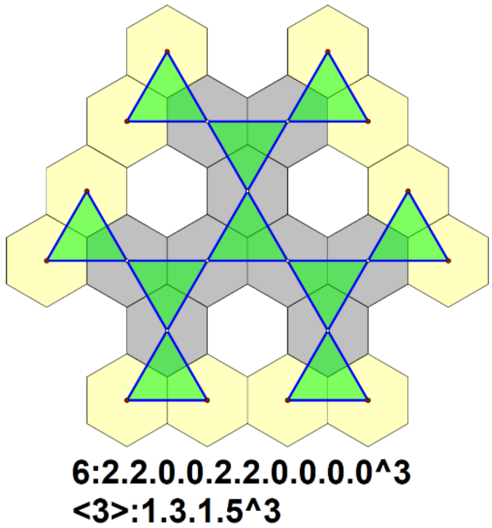

Figure 29: A polygon represented in both equilateral and equiangular polytile notation

Figure 29 shows this *equiangular polytile notation* as **<3>:1.3.1.5^3**, repeating these edges lengths 3 times to close.

5.1 Negative integer and serial-sided isogons

Equiangular polygons allow positive and negative turn angles. This is choice can be expressed in the sign of the integer edge length. So, for a <p/q> equiangular polygon, 1 means turn 360q/p degrees counterclockwise and move one unit, and -1 means turn 360q/p degrees clockwise and move one unit.

Figure 29 shows an example in a **serial-sided isogon** [19], which is an equiangular polygon with a sequence of lengths 1,2,3,4…n, that returns to the initial point at the end.

Note: Sequential edge lengths with the same sign can be given by ellipses (…).

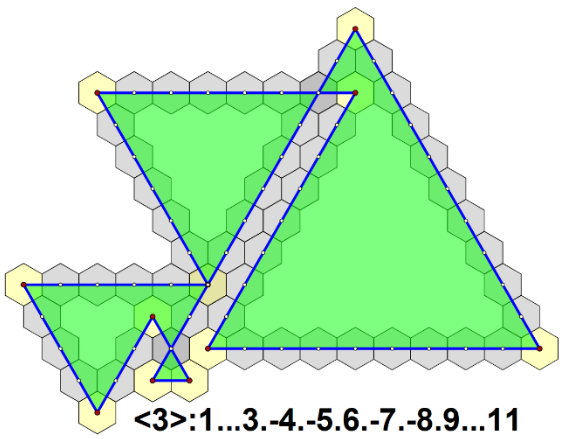

Figure 30: An equiangular polygon as a sequence of integers, a serial-sided isogon

5.2 Spirolaterals

Integer-length *equiangular polygons* have been explored in **spirolaterals** [8], polygons defined by a sequence of fixed angles and sequential edge lengths 1,2,3...*n* which repeat until the figure closes. Spirolaterals were invented and named by Frank C. Odds as a teenager in 1962, as square spirolaterals with 90° angles, drawn on graph paper, and later extended to 60° and 120°, and others based on rational divisors of 360. Odds used a notation $\mathbf{n_\theta}$ symbol for a simple spirolaterals with internal angles θ.

A spirolateral n_θ can be expressed in equiangular polytile notation as: **<p/q>:1.2.3….*n*^*c***, were $p/q=360/(180-\theta)$, and c is the number of cycles needed to return to the start. Figure 23 shows 5 simple examples, including one (lower left) that is a more general spirolateral, starting length 2.

Figure 31 shows a single spiral, <4>:1…9 and the 4 cycles that are needed to close it.

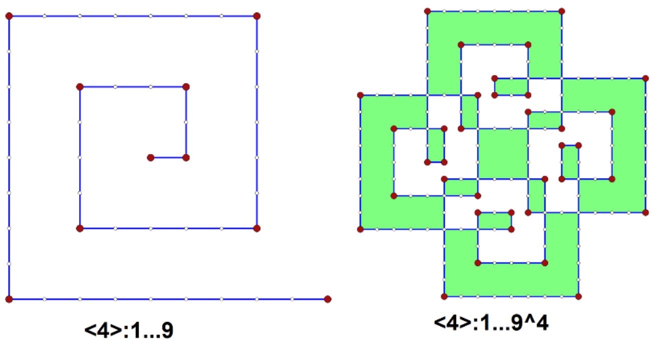

Figure 31: A single spiral and 4 cycles that close it

Figure 32 shows the first six 90° spirolaterals.
In equiangular polytile notation, the explicit cycle count is required. You can see $4_{90°}$ completes 4 turns with a translation, so no number of cycles will close it.

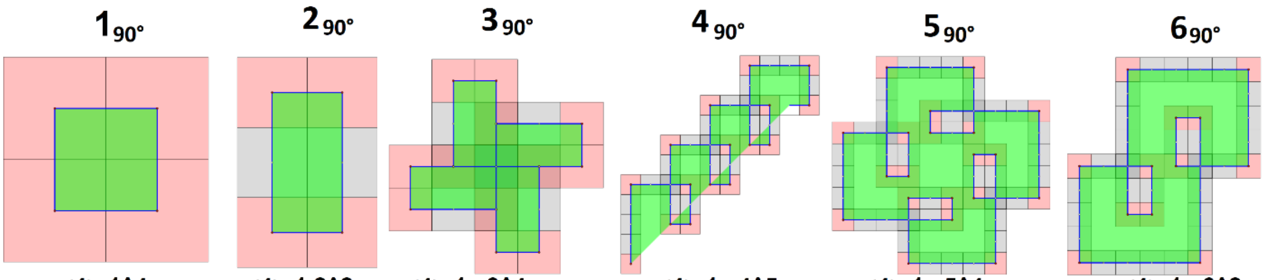

Figure 32: The first six simple 90° spirolaterals

Figure 33 shows five generalized spirolaterals with 5 levels, with 2 negative turns as superscripts.

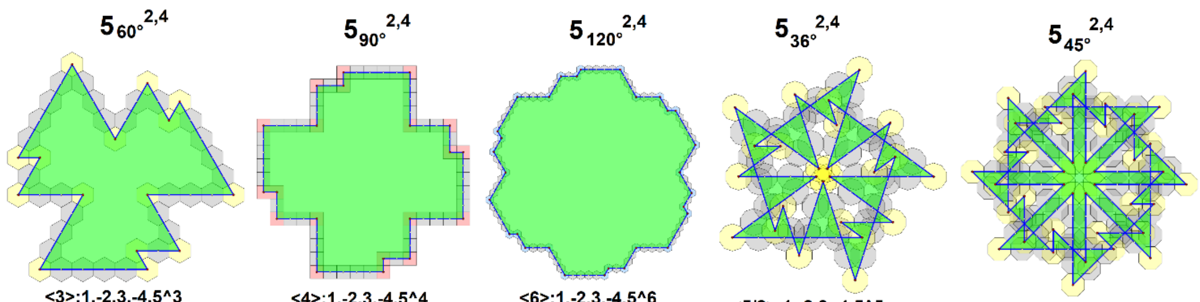

Figure 33: Five level 5 generalized spirolaterals, similar turns with varied angles

6. Improper polytiles

An **improper polytile** has most but not all equal/integer edges. There can be value in using the polytile notation to define these related polygons that are mostly equilateral. For example, improper tiles can be most useful for a tiling within a certain boundary.

6.1 Partial polytiles

A **partial polytile** is an open **polygonal chain** defined by polytile notation which is explicitly closed by adding an off-unit length edge.

A partial polytile is denoted as **p:a_1...a_{m-1}+** starting with the first vertex not on the non-unit edge and ending with a + to imply a last edge is added to close to the first vertex in the first edge. One vertex angle is unspecified, and the last angle given is not used in construction, but become whatever angle is necessary to return to the first vertex.

Figure 34 shows the **Type 15 Monohedral Pentagon** [3] which can self-tile the plane (left side), discovered in 2015. It has 3 equal edges, one double length edge, and one edge slightly off double length, seen in tile (a). The double-length edge can be represented by 2 colinear edges, 0 turn. With only one off-length edge, a partial polytile can represent it. Two chiral copies are needed, (b) and (c), starting on first ccw vertex not on the irregular edge. Each chiral copy can be seen as half of 2 polytiles (d), and (e), with cyclic and mirror symmetry.

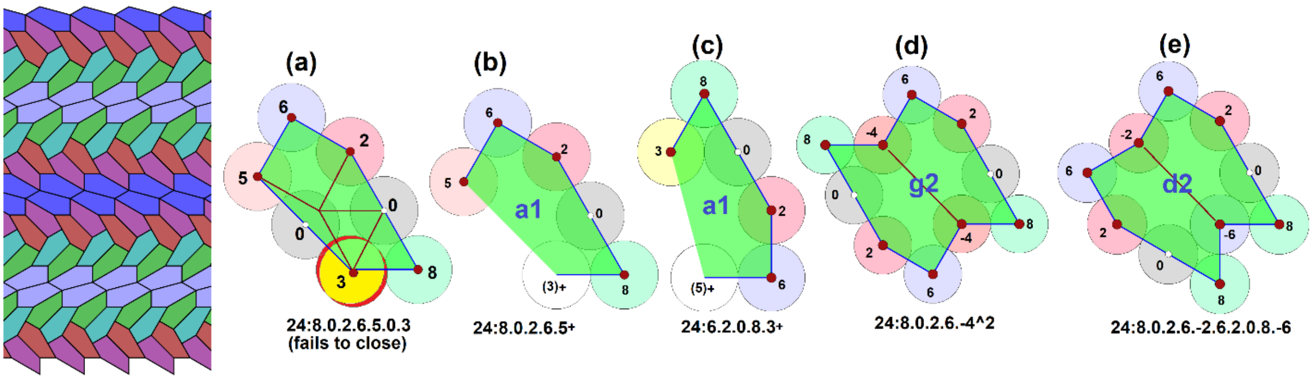

Figure 34: Type 15 pentagon, with 4 equal edges, existing as half of 2 decatiles

Partial polytiles can always become full tiles, by adding either a reflection across the odd edge or a 180-degree rotation copy. But they can also be combined with other partial tiles which have the same length. Figure 35 shows a regular decagon with 5 dissections at different vertices.

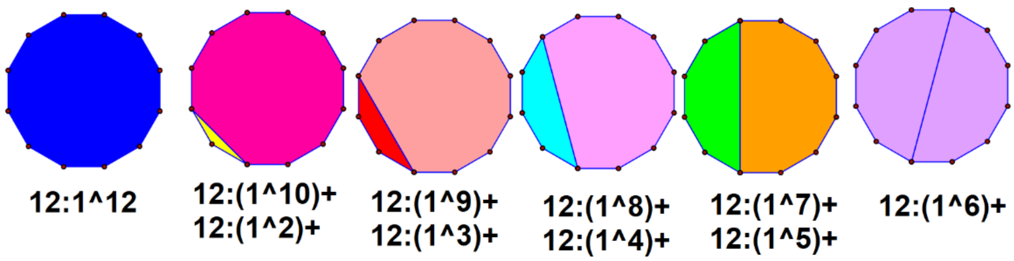

Figure 35: A regular dodecagon dissected into pairs of partial polytiles.

6.2 Fractional polytiles

A **fractional polytile** is a central dissection a polytile and adding 2 radial edges and the central point. Polytiles notation **is p:a_1…a_m^n/f**, with f as a divisor of *n*, starting at vertex a_1. This allows symmetric fractional polytiles like isosceles triangles, kite, dart, and other common and uncommon shapes of interest.

Figure 36 shows examples on double-sized hexagon **6:0.1^6**. There are eight different symmetry subdivisions depending on which vertex angle begins, ten with unique chirality. Figure 37 shows the same on an isotoxal hexagonal star. These subdivisions allow different colorings for each fraction.

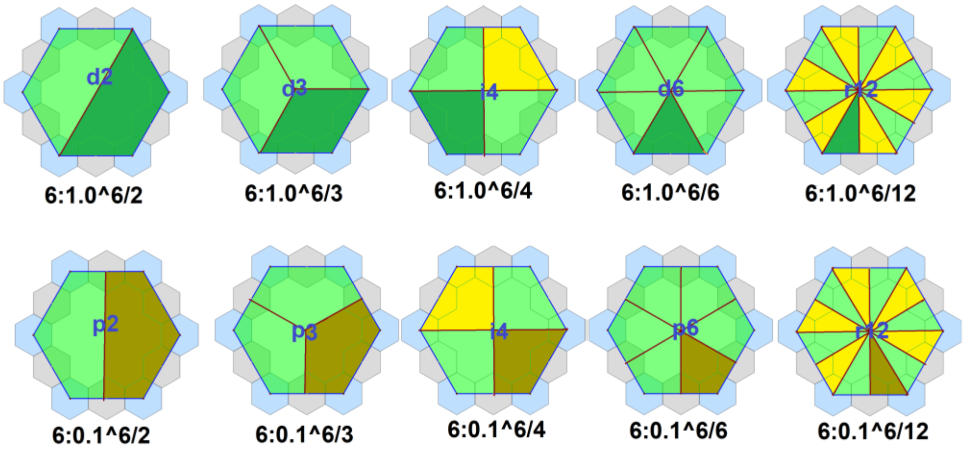

Figure 36: Ten symmetry fractional tiles from a regular hexagon

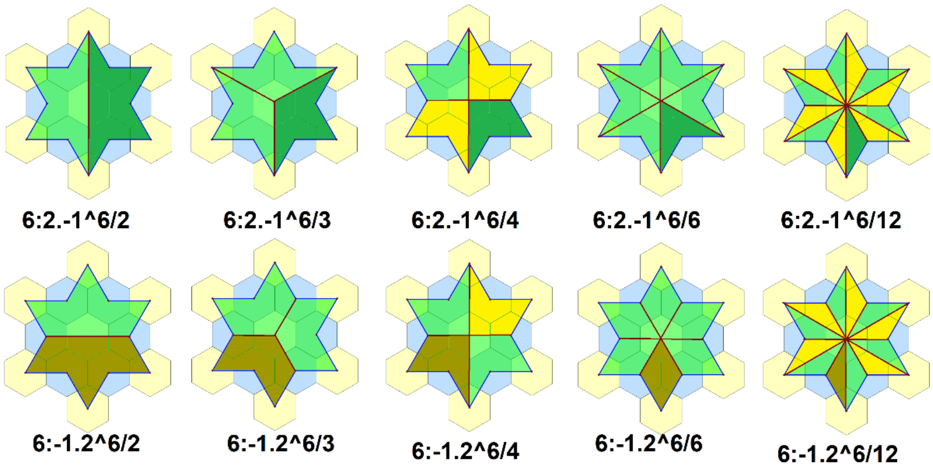

Figure 37: Ten symmetry fractional tiles from an isotoxal star

A future paper will explore these further. Tiling edge-to-edge with different edge lengths offers two possible interpretations:

1. Disallow connecting edge-to-edge unless edges are equal length.
2. Rescale a placed tile to match edge length which offers the possibility of recursively growing or shrinking tiles.

7. Online resources for exploration

This paper has demonstrated a wide breadth of possibilities in defining equilateral polygons and tilings that polytile notation can express. Future papers will expand in more details.

If you would like to explorer further now, more is available at the in-progress “Polytile Home” at https://bendwavy.org/polytile. Feedback is welcome!

There an interactive javascript program **Polytile Explorer** allows you to test out polytile notation and generate equilateral polygons, as well viewing galleries of examples.A second program will allow you to define polytiles and attach them together into edge-to-edge tessellations.

The homesite will also offer a polytile notation specification if anyone else would like to experiment with your own implementation.

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